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DYNAMIC STRESS CONCENTRATION IN GLASS-FIBER-REINFORCED PLASTIC

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INTRODUCTION

In this paper the problem of stress concentration around defects is investigated. The discrete model of glass-fiber-reinforced plastic [1, 2] is used, in which it is assumed that the fibers work on expansion and the bonding on shear; the inertia of both components is taken into consideration.

§1. Let the glass-fiber-reinforced plastic consist of an infinite number of fibers of width h with their number indicated by integers j. The fibers alternate with layers of the bonding of width H. The y axis is parallel and x axis perpendicular to the fibers. The displacement of the bonding along the y axis is denoted by $v_j(x, y, t)$; the index j shows that the point under investigation lies between the j-th and (j + 1)st fibers at a distance x from the j-th ($0 \leqslant x \leqslant H$); t is the time. The displacement of the fiber is denoted by $u_j(y, t)$. Hooke's law has the following form:

$$\sigma_j(y, t) = \mathrm{E}\partial u_j(y, t)/\partial y, \quad \tau_j(x, y, t) = G\partial v_j(x, y, t)/\partial x; \tag{1.1}$$

where σ_j and E are the normal stress and the Young modulus in the fiber and τ_j and G are the tangential stress and the shear modulus in the bonding.

It is shown in [2] that for zero initial conditions the behavior of the above system is described by the equations

$$\begin{gathered}\omega^2 \frac{d^2 u_j^L}{dy^2} + \beta^2 \left(u_{j-1}^L - \alpha u_j^L + u_{j+1}^L\right) = 0, \\ \beta^2 = G/\mathrm{E}, \ \omega^2 = Hh \operatorname{sh} \lambda/\lambda, \ \lambda = pH/c_2, \\ \alpha = p^2\omega^2/\beta^2 c_1^2 + 2\operatorname{ch}\lambda, \quad c_1^2 = \mathrm{E}/\rho_1, \ c_2^2 = G/\rho_2,\end{gathered} \tag{1.2}$$

where the index L denotes the Laplace (time) transform of the desired quantities, p is the transform parameter, and ρ_1 and ρ_2 are the densities of the fiber and bonding materials, respectively. After determining u_j^L, the displacement in the bonding is determined by the formula

$$v_j^L = \left[u_j^L(y, p) \operatorname{sh}(\lambda - \lambda x/H) + u_{j+1}^L(y, p) \operatorname{sh}(\lambda x/H)\right]/\operatorname{sh}\lambda, \tag{1.3}$$

and the stresses σ_j^L and τ_j^L are determined in accordance with (1.1) and (1.3). The solution of (1.2) that vanishes at $y \to \infty$ has the form [2]

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$$u_j^L(y,p)=\frac{1}{2\pi}\int_{-\pi}^{\pi} c(p,s)\exp\left(-\frac{\beta y}{\omega}\sqrt{\alpha-2\cos s}\right)e^{-isj}ds, \tag{1.4}$$

where c(p, s) is determined from the boundary conditions.

§2. Let the sample be stretched to infinity by a tension Q and at t = 0 in the cross section y = 0 let the fibers with numbers j = 0, ..., n suddenly break. Making use of the symmetry of the problem we take $y \geqslant 0$. We shall solve the obtained problem by subtracting the elastic field corresponding to homogeneous stretching by tension Q. For n = 0 this problem is solved in [3], the case of a simultaneous break of all fibers is investigated in [2], and the static problem is solved in [4].

According to the model adopted above (the bonding does not carry normal loads), the boundary conditions for system (1.2) have the following form:

$$\left.\frac{du_j^L}{dy}\right|_{y=0}=-\frac{Q}{Ep},\quad j=0,\ldots,n,\quad u_j^L|_{y=0}=0,\quad j\neq 0,\ldots,n. \tag{2.1}$$

It follows from (1.4) and (2.1) that only a finite number of Fourier coefficients are nonzero in the function c(p, s), i.e.,

$$c(p,s)=\sum_{k=0}^{n} c_k(p)\,e^{isk},$$

where $c_k(p)$ satisfies the system of linear algebraic equations

$$\sum_{k=0}^{n} c_k(p)\int_{-\pi}^{\pi} f(p,s)\,e^{is(k-j)}ds=-\frac{2\pi Q}{\beta pE},$$
$$f(p,s)=\left(\frac{p^2}{\beta^2 c_1^2}+\frac{2p}{hc_2}\,\frac{1-2\mu\cos s+\mu^2}{1-\mu^2}\right)^{1/2},\quad \mu=e^{-\lambda}. \tag{2.2}$$

In order to solve this system we expand c_k, f in Taylor series in the vicinity of the point $\mu = 0$:

$$c_k(p)=\sum_{m=0}^{\infty} c_{km}(p)\,\mu^m,\quad f(p,s)=\sum_{m=0}^{\infty} f_m(p,s)\,\mu^m.$$

Later on u_j^L, σ_j^L, and τ_j^L will also be determined in the form of power series of μ. According to the lag theorem of operational calculus, the terms of these series describe waves arriving from increasingly more distant fibers. We shall now proceed to solve the system. Putting $\mu = 0$ in (2.2), we obtain

$$\sum_{k=0}^{n} c_{k0}\int_{-\pi}^{\pi} f_0 e^{is(k-j)}ds=-2\pi Q/\beta pE.$$

However, $f_0(p, s)$ is independent of s, so that

$$\int_{-\pi}^{\pi} f_0 e^{is(k-j)}ds=\begin{cases}2\pi f_0, & k=j,\\ 0, & k\neq j\end{cases} \tag{2.3}$$

and the system under consideration breaks up into individual equations;

$$c_{j0}(p)=-Q/\beta pEf_0,\ j=0,\ldots,n,$$

We note that c_j is the transform of the displacement of the j-th fiber at y = 0. The free term of the Taylor series ($\mu = 0$, i.e., $H/c_2 = \infty$) describes the motion of a single fiber in infinite bonding. It is clear from physical considerations that before the arrival of the elastic waves from the adjacent fibers all the broken fibers move identically. Therefore, c_{j0} also does not depend on the number j.

We differentiate (2.2) with respect to μ and again put $\mu = 0$. We then have

$$\sum_{k=0}^{n} c_{k1} \int_{-\pi}^{\pi} f_0 e^{is(k-j)} ds + \sum_{k=0}^{n} c_{k0} \int_{-\pi}^{\pi} f_1 e^{is(k-j)} ds = 0.$$

Hence, in accordance with (2.3) we obtain

$$2\pi f_0 c_{j1} = -\sum_{k=0}^{n} c_{k0} \int_{-\pi}^{\pi} f_1 e^{is(k-j)} ds,$$

i.e., c_{j1} are expressed in terms of c_{k0} that are already determined. Proceeding in the same way we obtain c_{jm} in terms of c_{kl}, where $0 \leqslant l \leqslant m-1$. At each differentiation the corresponding system will break up by virtue of (2.3). This recurrent process has the following form:

$$c_{j0} = -Q/\beta p E f_0,$$
$$c_{jm} = \frac{1}{2\pi f_0} \sum_{k=0}^{m} \sum_{l=0}^{m-1} c_{kl} \int_{-\pi}^{\pi} f_{m-l} e^{is(k-j)} ds. \tag{2.4}$$

We now turn to the evaluation of $f_k = (1/k!)\partial^k f/\partial\mu^k|_{\mu=0}$. We rewrite f and f_0 in the form

$$f(p, s) = \sqrt{\frac{2p}{c_2 h}} \left(a + \frac{1 - 2\mu \cos s + \mu^2}{1 - \mu^2}\right)^{1/2}, \quad f_0 = \sqrt{\frac{2p}{c_2 h}} (a+1), \quad a = \frac{p c_2 h}{2\beta^2 c_1^2}.$$

Making use of the formula for obtaining the k-th derivative of a complex function ([5], formula 0.430) and assuming that the derivative is taken at the point $\mu = 0$, we obtain [taking $(-1)!! = 1$]

$$f_k(p, s) = \sqrt{\frac{2p}{c_2 h}} \sum_{l=1}^{k} \sum_{r=g(k-2l+1)}^{[k/2-l/2]} (-1)^{l+k-1} \times$$
$$\times \frac{(2l-3)!!\,(l+r-1)!\,(\cos s)^{2l+2r-k}}{r!\,(l-1)!\,(k-l-2r)!\,(2l+2r-k)!\,(a+1)^{l-1/2}}. \tag{2.5}$$

The square brackets denote that the integer part of the number is taken and $g(x) = \max(0\ [x/2])$. Since $2l + 2r - k \geqslant 0$, from formula 3.631.17 [5] we obtain

$$\int_{-\pi}^{\pi} (\cos s)^{2l+2r-k} e^{ims} ds = \frac{((-1)^k + (-1)^m)\,\pi}{2^{2l+2r-k}} \delta_0(2l + 2r - k - |m|)\, C_{2l+2r-k}^{r+l+m/2-k/2}, \tag{2.6}$$

where

$$\delta_0(x) = \begin{cases} 1, & \text{if} \quad x \geqslant 0, \\ 0, & \text{if} \quad x < 0; \end{cases}$$

C_k^l are the binomial coefficients. Substituting (2.5) and (2.6) into (2.4) we obtain

$$c_{j0} = -\frac{Q\sqrt{hc_2}}{2\beta E} \frac{1}{p^{3/2}(a+1)^{1/2}},$$
$$c_{jm} = \sum_{k=0}^{m} \sum_{l=0}^{m-1} \sum_{r=1}^{m-l} \sum_{q=g(m-l-2r+1)}^{[m-l-r/2]} \frac{(-1)^{r+m+l} + (-1)^{r+k+j}}{2^{\varkappa+1}} \times$$
$$\times \frac{\delta_0(\varkappa + 1 - |k-j|)\,(r+q-1)!\,(2r-3)!!}{(r-1)!\,q!\,(r-\varkappa)!\,[(\varkappa+k-j)/2]!\,[(\varkappa-k+j)/2]!} \frac{c_{kl}}{(a+1)^r}, \tag{2.7}$$
$$\varkappa = 2r + 2q + l - m.$$

It follows from (1.4), (1.1), and (2.7) that

$$u_j^L(0, p) = \frac{Q\sqrt{hc_2}}{2\beta E} \sum_{m=0}^{\infty} \mu^m \sum_{r=0}^{m} \frac{(Du)_{jmr}}{p^{3/2}(a+1)^{r+1/2}},$$
$$\sigma_{n+1}^L(0, p) = Q \sum_{m=0}^{\infty} \mu^m \sum_{r=0}^{m} \frac{(D\sigma)_{mr}}{p(a+1)^r}, \tag{2.8}$$

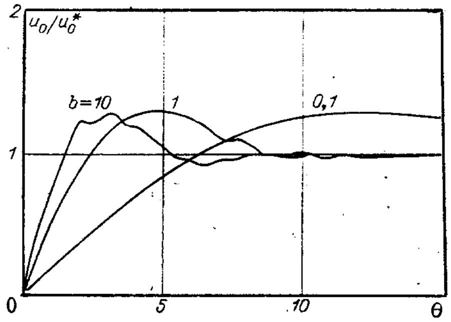

Fig. 1

$$\tau_n^L(0, 0, p) = -\frac{Q\beta\sqrt{h}}{2\sqrt{c_2}} \sum_{m=0}^{\infty} \mu^m \sum_{r=0}^{m} \frac{(D\tau)_{mr}}{p^{1/2}(a+1)^{r+1/2}},$$

where $(Du)_{jmr}$, $(D\sigma)_{mr}$, $(D\tau)_{mr}$ are some numerical coefficients which, unlike c_{jm}, no longer depend on p. Analogously to the formulas derived above, there are recurrent relationships for these coefficients also, which are not presented here due to their unwieldiness.

Using the inversion formulas of the Laplace transform from [6], we obtain

$$\frac{1}{p^{1/2}(a+1)^{r+1/2}} \doteqdot \sqrt{\frac{bc_2}{H}} \frac{2(2b\theta)^r}{\pi(2r-1)!!} \int_0^{\pi/2} e^{-b\theta\cos^2\varphi} \cos^{2r}\varphi d\varphi \equiv \sqrt{\frac{bc_2}{H}} J_r(b\theta),$$

$$\frac{1}{p(a+1)^r} \doteqdot \frac{1}{(r-1)!} \int_0^{b\theta} \xi^{r-1} e^{-\xi} d\xi \equiv \Phi_r(\xi),$$

$$\frac{1}{p^{3/2}(a+1)^{r+1/2}} \doteqdot \sqrt{\frac{2H}{bc_2}} 2b\theta \left(J_r(b\theta) - \frac{2b\theta}{2r-1} J_{r-1}(b\theta)\right),$$

where $b = 2\rho_2 H/\rho_1 h$ is the parameter of the material, which is equal to twice the ratio of the mass of one layer of bonding to the fiber mass; $\theta = tc_2/H$ is the dimensionless time. For the functions $J_r(\xi)$ and $\Phi_r(\xi)$ the following recurrent formulas hold:

$$J_0(\xi) = I_0(\xi/2)e^{-\xi/2}, \quad J_1(\xi) = \xi(I_0(\xi/2) - I_1(\xi/2))e^{-\xi/2},$$

$$J_r(\xi) = \frac{2\xi}{2r-1}\left(\left(1 + \frac{r-1}{\xi}\right) J_{r-1}(\xi) - J_{r-2}(\xi)\right),$$

$$\Phi_1(\xi) = 1 - e^{-\xi}, \quad \Phi_r(\xi) = \Phi_{r-1}(\xi) - \xi^{r-1}e^{-\xi}/(r-1)!,$$

where I_0 and I_1 are Bessel functions of imaginary argument and of zero and first order, respectively. We can now turn to (2.8) and write the displacement and the stress as functions of time and coordinates (only the tangential stress is given):

$$\tau_n(0, 0, \theta) = -\frac{\pi^{3/2}\beta Q b^{1/2}}{2^{n+2} n!} (2n+1)!! \sum_{m=0}^{\infty} \delta_0(\theta - m) \sum_{r=0}^{m} (D\tau)_{mr} J_r(b\theta - bm). \tag{2.9}$$

At any time t, (2.9) contains a finite number of terms, since the Heaviside function vanishes at sufficiently large t. Physically, it means that at any finite instant of time any fiber interacts with a finite number of other fibers. We shall give the expressions for the normal and tangential stresses for the case $n = 0$, $\theta < 4$:

$$\frac{\sigma_1(0, \theta)}{Q} = \left(\frac{1}{2} - \frac{1}{2} e^{-b(\theta-1)}\right) \delta_0(\theta - 1) + \left(-\frac{3}{16} + \frac{3}{16} e^{-b(\theta-3)} + \right.$$

$$\left. + \frac{13}{16} b(\theta - 3) e^{-b(\theta-3)} - \frac{5}{32} b^2(\theta - 3)^2 e^{-b(\theta-3)}\right) \delta_0(\theta - 3),$$

$$\frac{\tau_0(0, 0, \theta)}{\sqrt{b}\,\tau_0^*} = I_0(b\theta/2) e^{-b\theta/2} \delta_0(\theta) +$$

$$+ \left\{\left[2 - b(\theta - 2) + \frac{1}{6} b^2(\theta - 2)^2\right] I_0\left(\frac{b\theta - 2\theta}{2}\right) - \right.$$

$$\left. - \frac{1}{6} b(\theta - 2)[b(\theta - 2) - 5] I_1\left(\frac{b\theta - 2b}{2}\right)\right\} e^{-b(\theta-2)/2} \delta_0(\theta - 2)$$

(τ_0^* is the static value of the tangential stress).

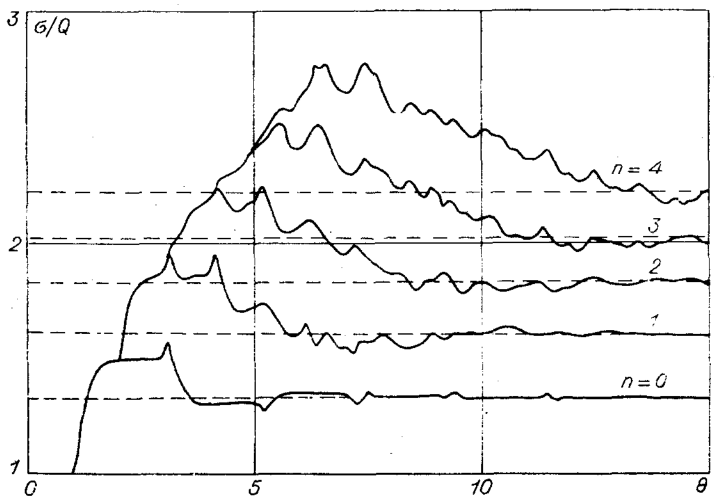

Fig. 2

The recurrence formulas for J_r and Φ_r and the coefficients $(D\tau)$ make formula (2.9) and analogous formulas for u_j and σ_{n+1} convenient for computations on a computer. The dependence of u_0/u_0^* (u_0^* is the static value of u_0) on the dimensionless time θ is shown in Fig. 1 for two broken fibers (graphs for a single broken fiber are given in [3]). The graphs of σ_{n+1}/Q for b = 10 are given in Fig. 2 for n = 0, 1, 2, 3, and 4, where unity, corresponding to homogeneous stretching by a tension Q at infinity, has been added to the values of this ratio. It is evident that the consideration of the inertia of the bonding results in an increase of the stress concentration at the first unbroken fiber compared to the static concentration (dashed lines); the ratio of the dynamic to static concentration is practically independent of b, has almost no dependence on n, and is equal to 1.2. The curves get closer together in dimensionless time as b increases. As $t \to \infty$, u_0 and σ_{n+1} tend to their static values.

The graphs of τ_n/τ_n^* for b = 1 are shown in Fig. 3 for one, two, and three broken fibers. The dynamic concentration of the tangential stress considerably exceeds the static concentration and their ratio increases with b (see graphs in [3]). It is evident that at the instants $\theta = 2m$, τ_n has discontinuities and in contrast to other quantities it does not tend to the static value with time (dashed lines). This fact, which seems strange at first sight, is a natural consequence of the idealization adopted. In the model the bonding represents a set of "strings" rigidly connected with the fibers and interacting with each other. The behavior of each "string" is described by the wave equation. In the n-th layer of the bonding, in the section y = 0, one end of the corresponding "string" is fixed, and, as can be shown, the velocity of the other end has a discontinuity at $\theta = 0$ and is continuous thereafter. Therefore, a discontinuity of the tangential stress that is constant in magnitude travels along the "string." This occurs not only in the n-th layer of the bonding but also everywhere on a crack. Discontinuities of the tangential stress will also occur in the bonding $y > 0$, but apparently the magnitude of the discontinuity will no longer be constant. It remains unclear whether τ_n will tend to its static value for $y > 0$.

On the basis of the above discussion we note the following:

1. All the other quantities for $y > 0$ can also be determined similarly. The coefficients c_k will be the same, since they are determined by the boundary conditions, while $f \exp(-\beta y f)$ will figure in all the operations instead of the function f.

2. The broken fibers need not necessarily be successive; it is possible to solve the problem with arbitrary arrangement of the broken fibers under the condition that all the fibers break on the line y = 0. It is also possible to consider nonsimultaneous breaking of different fibers. In this case the right-hand sides in system (2.2) will depend on the number of the equation, but the method of solving the system will remain the same.

3. It has been assumed so far that after the initial breaking of (n + 1) fibers the sample does not subsequently disintegrate. Let us consider at what instant and in what way the disintegration of the sample begins depending on the endurance characteristics of the fiber and bonding materials.

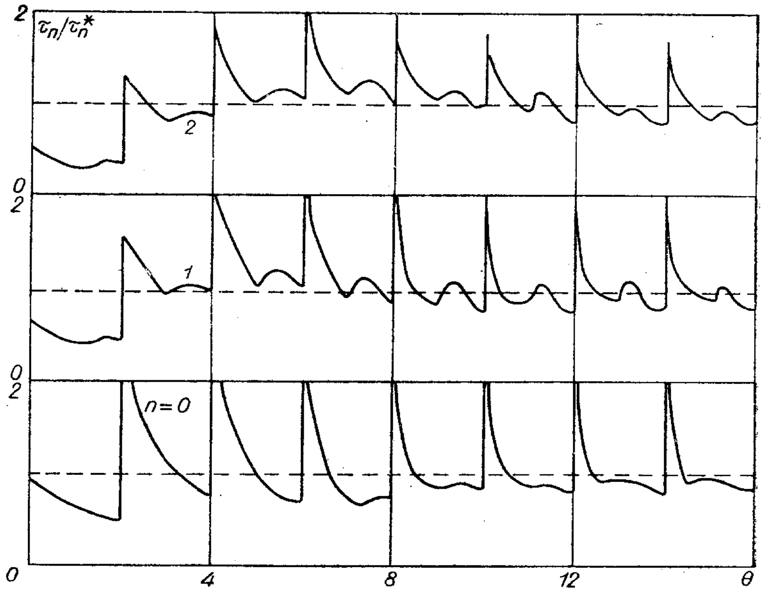

Fig. 3

We shall assume that the fiber-glass-reinforced plastic can disintegrate in two ways: by breaking of the fiber and by lamination of the bonding. Since the concentration of the normal stresses is maximum at the first unbroken fiber at y = 0, we shall assume that the breaking of the fiber may primarily occur just at this point. The concentration of the tangential stresses is maximum in the n-th layer of the bonding at the boundary of the (n + 1)-st or n-th fiber at y = 0; therefore, we shall assume that the peeling off of the bonding from the fiber will occur mainly at one of these points. If it is further assumed that the shearing strength of the bonding is roughly equal to the strength of its adhesion to the fiber, then the lamination of the bonding at any point within the sample can occur only after its peeling off from the fiber.

The model used here leads to finite stresses in the fibers and the bonding. Therefore, it may be possible to determine whether or not the sample will subsequently disintegrate depending on the normal strength of the fiber σ_* or the shearing strength of the bonding τ_* being exceeded. In the model of a continuous medium, infinite stresses are obtained at the tip of the fissure, and in the formulation of the problem of disintegration it is necessary to average the stresses in the vicinity of the crack over a certain length of the order of several interatomic distances; for example, the condition of equilibrium of Griffith's crack [7] can be obtained just in this way.

However, even in the present case the formulation of the conditions of strength according to the maximum stresses is not feasible, since the stresses change very rapidly with time (see Figs. 2 and 3) and the dynamic tangential stresses do not at all attenuate as $t \to \infty$ and do not tend to the static value.

Small changes in the model, for example, the consideration of the viscosity, may result in large changes in the maximum values of the stresses and cast doubt on the above inferences. Finally, it is not natural to set the behavior of a mechanical system as a function of conditions that occur only during a zero time interval (reaching the breaking point at once leads to disintegration). Therefore, we shall average the stresses over time in formulating the strength conditions. Thus, we take the strength conditions of the fiber and the bonding in the following form:

$$\frac{1}{t_*}\int\limits_{t-t_*}^{t}\sigma(\xi)\,d\xi<\sigma_*,\quad \frac{1}{t_*}\int\limits_{t-t_*}^{t}\tau(\xi)\,d\xi<\tau_*,$$

where t_* is the material constant having the significance of the time of averaging; σ_* and τ_* are, respectively, the strengths of the fiber and the bonding. These quantities must be determined experimentally. If σ and τ are constant in time, then (3.1) goes over into the ordinary static strength condition. The same situation will occur for sufficiently large t and t_* ($t > t_*$) if σ and τ tend to their static value with the increase in time even if only on the average. If $t_* \to 0$, then at the points of discontinuity of the integrands we find that the instantaneous values of the stresses need not exceed the breaking points.

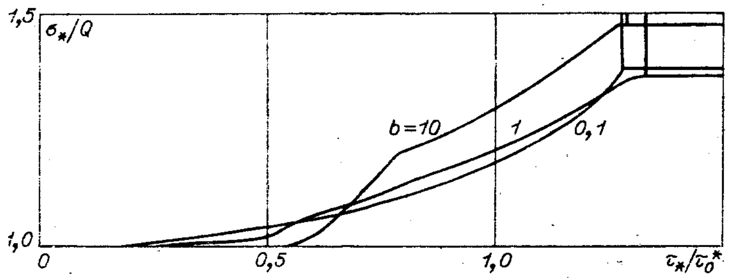

Fig. 4

Let the fiber-glass-reinforced plastic sample be stretched at infinity by a tension Q. It is natural to assume that $Q < \sigma_*$. Let one fiber suddenly break (simultaneous breaking of several adjacent fibers is improbable for $Q < \sigma_*$). The following four cases are possible: a) Two adjacent fibers break at a certain instant of time (from the symmetry of the problem); b) at a certain instant of time the bonding separates from the broken fiber or from adjacent fibers; c) events "a" and "b" occur simultaneously; d) subsequent disintegration does not occur. Which of these cases actually occurs depends on the value of the parameters σ_*/Q, τ_*/τ_0^*, b, and t_*. Thus, corresponding to each fiber-glass-reinforced plastic there is a point in the four-dimensional space with appropriate coordinates. The entire space can be divided into three regions corresponding to the cases "a," "b," and "d." The hypersurface separating the first and the second regions will correspond to case "c." The boundaries of these regions for b = 0.1, 1, 10 are shown in Fig. 4 for t_* equal to two transit times of the shear wave between the fibers. The points corresponding to samples disintegrating by breaking of fibers lie below the boundary line and those corresponding to samples disintegrating by the separation of the bonding lie above. The points corresponding to nondisintegrating samples lie in the upper right part of the graph. The boundaries of the regions practically do not change with the change in b. This is due to the averaging of the dynamic stresses over time, since the instantaneous concentration of τ has a strong dependence on b. Computations carried out for t_* equal to five transit times showed that the position of the boundary has a weak dependence on t_* also (the changes in the boundary on varying t_* are of the same order as on varying b).

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